

Fock Space Decomposition of Lévy Processes

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Abstract

We show that the general Levy process can be embedded in a suitable Fock space, classified by cocycles of the real line regarded as a group. The formula of de Finetti corresponds to coboundaries. Kolmogorov's processes correspond to cocycles of which the derivative is a cocycle of the Lie algebra of \mathbf{R} . The Lévy formula gives the general cocycle.

1 Cyclic representations of groups

Let G be a group and $g \mapsto U_g$ a multiplier cyclic representation of G on a Hilbert space \mathcal{H} , with multiplier $\sigma : G \times G \rightarrow \mathbf{C}$ and cyclic vector Ψ . This means that

- $U_g U_h = \sigma(g, h) U_{gh}$ for all $g, h \in G$.
- $u(e) = I$ where e is the identity of the group and I is the identity operator on \mathcal{H} .
- $\text{Span} \{U_g \Psi : g \in G\}$ is dense in \mathcal{H} .

If $\sigma = 1$ we say that U is a true representation.

Recall that a multiplier of a group G is a measurable two-cocycle in $Z^2(G, U(1))$; so σ is a map $G \times G \rightarrow U(1)$ such that $\sigma(e, g) = \sigma(g, e) = 1$ and

$$\sigma(g, h) \sigma(g, hk)^{-1} \sigma(gh, k) \sigma(h, k)^{-1} = 1. \quad (1)$$

(1) expresses the associativity of operator multiplication of the $U(g)$. σ is a coboundary if there is a map $b : G \rightarrow U(1)$ with $b(e) = 1$ and

$$\sigma(g, h) = b(gh)/(b(g)b(h)). \quad (2)$$

We also need the concept of a one cocycle ψ in a Hilbert space \mathcal{K} carrying a unitary representation V . ψ is a map $G \rightarrow \mathcal{K}$ such that

$$V(g)\psi(h) = \psi(gh) - \psi(g) \quad \text{for } g, h \in G. \quad (3)$$

ψ is a coboundary if there is a vector $\psi_0 \in \mathcal{K}$ such that

$$\psi(g) = (V(g) - I)\psi_0. \quad (4)$$

Coboundaries are always cocycles. We say that, in (2) and (4), σ is the coboundary of b and ψ is the coboundary of ψ_0 .

We say that two cyclic σ -representations $\{\mathcal{H}, U, \Psi\}$ and $\{\mathcal{K}, V, \Phi\}$ are *cyclically equivalent* if there exists a unitary operator $W : \mathcal{H} \rightarrow \mathcal{K}$ such that $V_g = WU_gW^{-1}$ for all $g \in G$, and $W\Psi = \Phi$. Any cyclic multiplier representation $\{\mathcal{H}, U, \Psi\}$ defines a function F on the group by

$$F(g) := \langle \Psi, U_g \Psi \rangle, \quad (5)$$

which satisfies σ -positivity:

$$F(e) = 1 \quad (6)$$

$$\sum_{ij} \bar{\lambda}_i \lambda_j \sigma(g_i^{-1}, g_j) F(g_i^{-1} g_j) \geq 0. \quad (7)$$

F is called the characteristic function of the representation, because

- Two cyclic multiplier representations of G are cyclically equivalent if and only if they have the same characteristic function;
- Given a function on G satisfying σ -positivity, then there exists a cyclic σ -representation of which it is the characteristic function.

If $G = \{s \in \mathbf{R}\}$, $\sigma = 1$ and U_s is continuous, then F obeys the hypotheses of Bochner's theorem and defines a probability measure μ on \mathbf{R} . More generally, we can apply Bochner's theorem (if $\sigma = 1$) to any one-parameter subgroup $s \mapsto g(s) \in G_0 \subseteq G$. Then $U_{g(s)}$, $s \in \mathbf{R}$ is a one-parameter unitary group; its infinitesimal generator is a self-adjoint operator X on \mathcal{H} . The relation to μ is given as follows: let $X = \int \lambda dE(\lambda)$ be the spectral resolution of X . Then

$$\mu(\lambda_1, \lambda_2] = \langle \Psi, (E(\lambda_2) - E(\lambda_1)) \Psi \rangle. \quad (8)$$

Conversely, given any random variable X on a probability space (Ω, μ) , we can define the cyclic unitary representation of the group \mathbf{R} by the multiplication operator

$$U(s) = \exp\{isX\} \quad (9)$$

and use the cyclic vector $\Psi(\omega) = 1$ on the Hilbert space $L^2(\Omega, d\mu)$. In this way, probability theory is reduced to the study of cyclic representations of abelian groups, and quantum probability to the study of cyclic σ -representations of non-abelian groups.

2 Processes as Tensor Products

Given a cyclic σ -representation $\{\mathcal{H}, U, \Phi\}$ of a group G , we can get a multiplier representation of the product group $G^n := G \times G \times \dots \times G$ (n factors) on $\mathcal{H} \otimes \mathcal{H} \dots \otimes \mathcal{H}$, by acting on the vector $\Psi \otimes \Psi \dots \otimes \Psi$ by the unitary operators $U(g_1, \dots, g_n) := U(g_1) \otimes \dots \otimes U(g_n)$, as each g_j runs over the group G . The resulting cyclic $\sigma^{\otimes n}$ -representation is denoted

$$\{\mathcal{H}^{\otimes n}, U^{\otimes n}, \Psi^{\otimes n}\}. \quad (10)$$

The twisted positive function on G^n defined by this cyclic representation is easily computed to be

$$F^{\otimes n}(g_1, \dots, g_n) = F(g_1)F(g_2) \dots F(g_n). \quad (11)$$

If G has a one-parameter subgroup G_0 , then the infinitesimal generators X_j of this subgroup in the j^{th} place define random variables ($j = 1, \dots, n$) that are all independent in the measure $\mu^{\otimes n}$ on \mathbf{R}^n

defined by $F^{\otimes n}$, and are all identically distributed. They can thus be taken as the increments of a process in discrete time $t = 1, \dots, n$. To get a process with time going to infinity, we can embed each tensor product $\mathcal{H}^{\otimes n}$ in the “incomplete infinite tensor product” of von Neumann, denoted

$$\bigotimes_{j=1}^{j=\infty} {}^\Psi \mathcal{H}_j \quad \text{where } \mathcal{H}_j = \mathcal{H} \text{ for all } j. \quad (12)$$

It is harder to construct processes in continuous time. We made [12] the following definition:

Definition 1 A cyclic G -representation $\{\mathcal{H}, U, \Psi\}$ is said to be *infinitely divisible* if for each positive integer n there exists another cyclic G -representation $\{\mathcal{K}, V, \Phi\}$ such that $\{\mathcal{H}, U, \Psi\}$ is cyclically equivalent to $\{\mathcal{K}^{\otimes n}, V^{\otimes n}, \Phi^{\otimes n}\}$.

The picturesque notation $\{\mathcal{H}^{\otimes \frac{1}{n}}, U^{\otimes \frac{1}{n}}, \Psi^{\otimes \frac{1}{n}}\}$ can be used for $\{\mathcal{K}, V, \Phi\}$.

If $G = \mathbf{R}$ then $\{\mathcal{H}, U, \Psi\}$ is infinitely divisible if and only if the corresponding measure μ given by Bochner’s theorem is infinitely divisible [12]. It is clear that $\{\mathcal{H}, U, \Psi\}$ is infinitely divisible if and only if there exists a branch of $F(g)^{\frac{1}{n}}$ which is positive semi-definite on G .

This criterion was extended in [9] to σ -representations. In that case, for each n , there should exist an n^{th} root $\sigma(g, h)^{\frac{1}{n}}$ which is also a multiplier. One can then consider cyclic representations such that for each n , $F(g)^{\frac{1}{n}}$ has a branch which is $\sigma^{\frac{1}{n}}$ -positive semi-definite.

If $\{\mathcal{H}, U, \Psi\}$ is an infinitely divisible G -representation, then we may construct a continuous tensor product of the Hilbert spaces \mathcal{H}_t , where $t \in \mathbf{R}$ and all the Hilbert spaces are the same. This gives us, in the non-abelian case, quantum stochastic processes with independent increments. See references. The possible constructions are classified in terms of cocycles of the group G . Here we shall limit discussion to the analysis of the Lévy formula in these terms.

3 The cocycle

Let $F : G \rightarrow \mathbf{C}$ and $F(e) = 1$. It is a classical result for $G = \mathbf{R}$ that a function $F^{\frac{1}{n}}$ has a branch that is positive semidefinite for all $n > 0$ if and only if $\log F$ has a branch f such that $f(0) = 0$ and f is *conditionally* positive semidefinite. This is equivalent to $f(x - y) - f(x) - f(-y)$ being positive semidefinite. This result is easily extended to groups [12] and σ -representations [9, 10]. Let us consider the case where $\sigma = 1$. It follows that an infinitely divisible true cyclic representation $\{\mathcal{H}, U, \Psi\}$ of G defines a conditionally positive semidefinite function $f(g) = \log \langle \Psi, U(g) \Psi \rangle$, so that

$$\sum_{j,k} \bar{\alpha}_j \alpha_k \left(f(g_j^{-1} g_k) - f(g_j)^{-1} - f(g_k) \right) \geq 0. \quad (13)$$

We can use this positive semidefinite form to make $\text{Span } G$ into a pre-scalar product space, by defining

$$\langle \psi(g), \psi(h) \rangle := f(g^{-1}h) - f(g^{-1}) - f(h), \quad g, h \in G. \quad (14)$$

Let \mathcal{K} be the Hilbert space, that is the separated and completed space got this way. There is a natural injection $\psi : G \rightarrow \mathcal{K}$, namely, $g \mapsto [g]$, the equivalence class of g given by the relation $g \sim h$ if the seminorm defined by (13) vanishes on $g - h$. The left action of the group G on this function is not quite unitary; in fact the following is a unitary representation [1]:

$$V(h)\psi(g) := \psi(hg) - \psi(h). \quad (15)$$

One just has to check from (14) that the group law $V(g)V(h) = V(gh)$ holds, and that

$$\langle V(h)\psi(g_1), V(h)\psi(g_2) \rangle = \langle \psi(g_1), \psi(g_2) \rangle. \quad (16)$$

Thus we see that $\psi(g)$ is a one-cocycle relative to the G -representation V [1].

4 The embedding theorem

Given a Hilbert space \mathcal{K} , the *Fock space* defined by \mathcal{K} is the direct sum of all symmetric tensor products of \mathcal{K} ,

$$EXP \mathcal{K} := \mathbf{C} \oplus \mathcal{K} \oplus (\mathcal{K} \otimes \mathcal{K})_s \oplus \dots \quad (17)$$

The element $1 \in \mathbf{C}$ is called the Fock vacuum. The following *coherent states* form a total set in $EXP \mathcal{K}$:

$$EXP \psi := 1 + \psi + (1/2!)\psi \otimes \psi + \dots, \quad \psi \in \mathcal{K}. \quad (18)$$

The notation is natural, in view of the easy identity

$$\langle EXP \psi(g), EXP \psi(h)h \rangle = \exp\{\langle \psi(g), \psi(h) \rangle\}. \quad (19)$$

Then the embedding theorem [12] says that if $\{\mathcal{H}, U, \Psi\}$ is an infinitely divisible cyclic representation, then it is cyclically equivalent to the cyclic representation W on $EXP \mathcal{K}$, with the Fock vacuum as the cyclic vector, with the unitary representation $W(h)$ defined on the total set of coherent states by

$$W(h)EXP \psi(g) = F(hg)/F(g)EXP \psi(hg). \quad (20)$$

The proof is simply a verification. This result has been called [12, 9] the Araki-Woods embedding theorem; more properly this name belongs to the embedding [12] of the *process* that one constructs from $\{\mathcal{H}, U, \Phi\}$, which is similar to a deep result in [2]. For the group bfr it amounts to the Wiener chaos expansion.

5 The Lévy Formula

Although the above theory was developed for quantum mechanics, it includes the theory of Lévy processes, which is the class of processes with independent increments. This is just the case when the group in question is \mathbf{R} , or \mathbf{R}^n . The latter group has projective representations in $n > 1$, and using these leads to the free quantised field [11]. The true representations lead to generalised random fields.

Every projective representation of \mathbf{R} is a true representation, which is multiplicity free if it is cyclic. By reduction theory, it is then determined by a measure on the dual group, here \mathbf{R} . Araki showed that a one-cocycle can be algebraic or topological. For the group \mathbf{R} , the algebraic cocycles are all of the form $f(x-y) - f(x) - f(-y) = axy$. This is satisfied by the Gaussian term $\log F(x) = -\frac{a}{2}x^2 + ibx$, and this is the only possibility. The Poisson(λ) is an example of a coboundary, when $\log F(t) = c\lambda(e^{ipt} - 1)$ for some p , the increment of the jumps. The weighted mixture of these coboundaries gives di Finetti's formula [3]:

$$\log F(t) = \lambda \left\{ ibt - \frac{a^2 t^2}{2} + c \int (e^{ipt} - 1) dP(p) \right\}. \quad (21)$$

That this is not the most general infinitely divisible measure was recognised by Kolmogorov [7]. In our terms, this is the statement that not all topological cocycles are coboundaries (the cohomology

is non-trivial). Kolmogorov considered random variables with finite variance relative to the measure dP . This is equivalent in our terms to $dP = |\hat{\psi}(p)|^2 dp$ and the cocycle ψ being of the form $\psi(x) = (V(x) - I)\psi_0$, where $i\partial_x\psi_0$ is square integrable over the group \mathbf{R} , but ψ_0 might not be. Thus, ψ is a cocycle for the Lie algebra of the group, a case treated in [14]. This gives us Kolmogorov's formula

$$\begin{aligned} \log F(t) = & \lambda \left\{ ibt - a^2 t^2 / 2 + \right. \\ & \left. + \int \left(e^{ipt} - 1 - ipt \right) |\psi(p)|^2 dp \right\}. \end{aligned} \quad (22)$$

The term $\int (-ipt) |\psi(p)|^2 dp$ is possibly divergent near $p = 0$ but is not required to exist on its own near $p = 0$, since the function $M = e^{ipt} - 1 - ipt$ behaves as p^2 near the origin. But to retain a meaning, Kolmogorov's formula does need $p|\hat{\psi}(p)|^2$ to be integrable at infinity. This is not needed for the general cocycle, so the formula is not the most general.

Lévy gave the answer [8] by replacing M by

$$e^{ipt} - 1 - ipt/(1 + p^2), \quad (23)$$

so that $\hat{\psi}$ has no constraint at infinity other than being L^2 . The general form of an infinitely divisible random variable, the Lévy formula, in effect constructs the most general cocycle of the group \mathbf{R} by requiring only that $p\hat{\psi}(p)$ should be locally square-integrable at $p = 0$, and $\hat{\psi}(p)$ should be square-integrable at all other points.

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